

Comment on “Chaotic orbits for spinning particles in Schwarzschild spacetime”

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The astrophysical relevance of chaos for a test particle with spin moving in Schwarzschild spacetime was the objective of [1]. Even if the results of the study seem to be qualitatively in agreement with similar works, the study presented in [1] suffers both from theoretical and technical issues. These issues are discussed in this comment.

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1. THE ISSUE OF SCALING

The Mathisson-Papapetrou (MP) equations [2, 3] describe the motion of an extended body in the pole-dipole approximation on a curved spacetime. In the MP description the body has some internal degrees of freedom, which are constraint to fix its centroid, i.e. the wordline along which the body moves. This constraint is imposed when a spin supplementary condition (SSC) is chosen. In [1] the Tulczyjew (T) SSC has been chosen. This makes [1] comparable with previous similar works since T SSC has been used in [5] (Schwarzschild background as in [1]) and in [6–8] (Kerr background).

When studying the MP equations with T SSC one chooses the mass of the test particle to be described by the contraction of the four-momentum, i.e., $P^a P_a = -\mathcal{M}^2$, since this mass is a conserved quantity for T SSC (see, e.g., [9]). In [1] the mass is chosen with respect to the four-velocity, i.e. $P^a V_a = -\mu$, which is not a conserved quantity for T SSC (see, e.g., [9]). Choosing the one mass over the other is a matter of what one sets as an observer. Since in T SSC we observe with respect to the four-momentum P^a is somehow not “natural” to measure the mass with respect to V^a . Apart from being conceptually strange to choose μ as the mass for the MP with T SSC, this brings along some further complications when the spin is scaled with respect to μ m as stated to be done in [1], where m is the mass of the central black hole (the notation of [1] is adopted). For example, the measure of the spin is a constant of motion for T SSC (see, e.g., [9]), but when one normalizes the spin with something that varies, the constancy of the spin ceases to be the case.

In order to understand these complications, let us discuss the scaling issue in more detail. The spin is usually scaled with respect to $m \mathcal{M}$, or with respect to \mathcal{M}^2 to make it a dimensionless quantity (see, e.g., discussion in [7]). The MP equations can be written in scale free units

if we use the $m \mathcal{M}$ scaling for spin as

$$\begin{aligned} \frac{D P^\mu / \mathcal{M}}{d\tau / m} &= -\frac{1}{2} (R^\mu{}_{\nu\kappa\lambda} m^2) V^\nu \frac{S^{\kappa\lambda}}{m \mathcal{M}} , \\ \frac{D S^{\mu\nu} / (m \mathcal{M})}{d\tau / m} &= (P^\mu V^\nu - V^\mu P^\nu) / \mathcal{M} , \end{aligned} \quad (1)$$

where each quantity was written with respect to its scale factor. It is easy to see that the scale factors cancel out. Now, if we follow the scalings suggested in [1], then

$$\begin{aligned} \frac{D P^\mu / \mathcal{M}}{d\tau / m} &= -\frac{1}{2} (R^\mu{}_{\nu\kappa\lambda} m^2) V^\nu \frac{S^{\kappa\lambda}}{m \mu} , \\ \frac{D S^{\mu\nu} / (m \mu)}{d\tau / m} &= (P^\mu V^\nu - V^\mu P^\nu) / \mathcal{M} , \end{aligned} \quad (2)$$

and we get

$$\begin{aligned} \frac{D P^\mu}{d\tau} &= -\frac{\mathcal{M}}{2\mu} R^\mu{}_{\nu\kappa\lambda} V^\nu S^{\kappa\lambda} , \\ \frac{D S^{\mu\nu}}{d\tau} &= \frac{\mu}{\mathcal{M}} (P^\mu V^\nu - V^\mu P^\nu) , \end{aligned} \quad (3)$$

where the scales do not vanish.

One could argue that the scales would vanish if the momentum P^a was scaled with respect to μ and not with respect to \mathcal{M} . This is true, but in [1] it is said that $P^a P_a = -1$, which suggests either that the momentum in [1] is scaled with respect to \mathcal{M} or that $P^a P_a = -\mathcal{M}^2 / \mu^2 = -1$. The latter cannot be the case since during the evolution μ varies, while \mathcal{M} is a constant, and in general $\mu \neq \mathcal{M}$ for T SSC (see, e.g., [9]). The rescaling of the momentum with respect to \mathcal{M} is reflected on the eqs. (10), (13), (14) in [1]. What is missing from eqs. (13), (14) is the rescaling of the spin four-vector S^a . S^a is the vector counterpart of S^{ab} see, e.g., eq. (10) in [1]. Eqs. (13), (14) hold for the $m \mathcal{M}$ rescaling of the spin (e.g., the usual rescaling for T SSC used in [6]). But, if the $m \mu$ rescaling of the spin was used in [1], as stated in Sec. II of [1], then the corresponding eqs. (13), (14) in [1] should include the ratios μ / \mathcal{M} as shown in the corresponding eqs. (3) shown above. Thus, the rescaling implied by eqs. (13), (14) in [1] is inconsistent with the $m \mu$ rescaling of the spin stated in [1].

This inconsistency is reflected also on the eqs. (15–17) of [1]. Furthermore, in eq. (15) of [1] the first term P^a

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in the numerator should not share the denominator with the second term. Namely, eq. (15) should read

$$V^\mu = \frac{\mu}{\mathcal{M}^2} \left(P^\mu - \frac{{}^*R^{\nu\rho\kappa\lambda} S_\rho P_\kappa S_\lambda}{\mathcal{M}^2 + {}^*R^{\alpha\beta\gamma\delta} S_\alpha P_\beta S_\gamma P_\delta} \right). \quad (4)$$

But, this is probably just a typo. The main issue here is that the stated rescaling is in contradiction with the formulas presented.

Note also that if μ was considered constant, we would not be able to normalize the four-velocity so that $V_a V^a = -1$ in order to evolve the MP equations with T SSC. The variability of μ is what allows the four-velocity normalization (see, e.g., [9]).

2. THE POINCARÉ SECTION AND THE LYAPUNOV NUMBER ISSUE

Poincaré sections are a useful tool to discern chaos from order in a two degrees of freedom Hamiltonian system. Such a Hamiltonian system, for instance, corresponds to geodesic motion in an axisymmetric and stationary spacetime background. Regular orbits are represented by closed zero width smooth curves, while chaotic orbits are represented by scattered points covering a non-zero width space on the section. So, the Poincaré sections shown in Figs. (1), (2) of [1] should indeed represent regular orbits.

However, the MP equations with T SSC have not been yet been described by a canonical Hamiltonian formalism (contrary to what is stated on page 3 in [1]), and the spin introduces more phase space dimensions to the system than the geodesic spinless case. So, one should be careful when interpreting 2D “Poincaré” sections for a test particle with spin moving on a Schwarzschild background. In the latter system what one gets usually for regular orbits on a 2D section are projections of tori which dimensionality is higher than three. These projections on a 2D section are represented by no zero width curves in the case of regular orbits, and there is no straightforward way to discern chaos from order just from inspecting these 2D sections. Thus, from Figs. (3), (4) of [1] one cannot tell whether the orbits are regular or not contrary to what is stated in [1] (see, e.g., caption of Fig. 4 in [1]). Actually, in [8] there is a case (Fig. 3 in [8]), where an orbit looking like those in Figs. (3), (4) of [1] was characterized as “chaotic mimic”, because when such 2D section was tested with other indicators of chaoticity, the other indicators implied that the orbit was regular.

One such indicator of chaoticity is the characteristic Lyapunov number λ . Lyapunov numbers are used to cross check the results of Figs. (3), (4) in [1]. However, defining the Lyapunov number for curved spacetimes is not a straightforward task as discussed in Sec. III of [1]. There is the issue of the time, and of the deviation vector ξ . The proper time τ , used in [1], is the time usually employed for trajectories in curved spacetime to solve the

time issue. But, the deviation vector is a more complicated problem. This vector is defined in a tangent space of the phase space. More precisely this tangent space is a collection of tangent spaces along the points consisting the trajectory. There is even a difficulty to comprehend what exactly this tangent space means in the general relativistic setup. But apart from the latter issue, the main problem with the deviation vector is what is the norm of this vector. Different norms $\xi = |\xi|$ result in different values for the Lyapunov number. It was found, however, that the sign of the Lyapunov number should not be affected by the choice of the norm. In [1] it is stated that for simplicity the Euclidean norm was preferred as the norm for ξ in their work, but no further informations about the explicit form of the norm are provided. Namely, there are the questions of how was the Euclidean norm applied in the Schwarzschild coordinates of the orbit, and how was the spin incorporated in the Euclidean norm of the deviation vector. Without the above explanations the results of this work are not reproducible and ambiguous.

A standard way to find whether an orbit is chaotic or not by using Lyapunov numbers is the $\ln \lambda$ vs. $\ln \tau$ plot (see, e.g., [8]). For a regular orbit the deviation vector grows linearly, i.e. $\xi \propto \tau$, which means that $\lambda \propto \frac{\ln \tau}{\tau}$. On the logarithmic plot this implies that for a regular orbit $\lim_{\tau \rightarrow 0} \ln \lambda \rightarrow -\infty$, i.e. $\lambda \rightarrow 0^+$, with a slope equal to -1 . Note that the Lyapunov number is practically evaluated for finite time, and during this even for regular orbits $\lambda > 0$. As long as $1/\lambda$ is of the order of magnitude of the time τ we have evolved the orbit, we cannot tell whether an orbit is chaotic or not. For a chaotic orbit the deviation vector grows exponentially $\xi \propto e^{\lambda \tau}$, which means that in the logarithmic plot we get a constant value $\ln \lambda$ after the Lyapunov time $\tau_\lambda = \lambda^{-1}$ is reached. In order to be sure that one gets a chaotic orbit, one has to evolve the orbit at least for two orders of magnitude more after τ_λ is reached so λ is not any more comparable with $1/\tau$.

In [1] the above standard procedure is absent. The procedure to find the Lyapunov number in [1] is based on a phenomenological model (eq (28) in [1]) which is irrelevant with the basic principles describing the evolution of the deviation vector discussed in the above paragraph. The example in Fig. 6 of [1] evolves an orbit for $\tau = 10^5$ and predicts an orbit with a Lyapunov number $\lambda \approx 3.787 \cdot 10^{-4}$. For $\tau = 10^5$ $\lambda \propto \frac{\ln \tau}{\tau} \approx 6.47 \cdot 10^{-4}$. Namely, for the amount of time the orbit has been evolved in Fig. 6 the Lyapunov number has a value that is comparable with a value of λ corresponding to a regular orbit. Thus, one cannot tell safely whether the orbit is chaotic or not. In fact, there are many Lyapunov-like chaotic indicators (see, e.g., [10] for a review), but none indicator can safely reveal the chaotic nature of an orbit at time scales comparable with the Lyapunov time. All the indicators show the nature of the orbit, much after the magnitude of the Lyapunov time has been reached.

In order to investigate the dependence of the chaoticity of the MP equations on the spin's value in [1], the

energy, the angular momentum, the initial radius r of the orbit and the orientation of the spin are kept constant, while the spin's value varies (Figs. 7–13, 15 in [1]). This might seem reasonable since the investigation depends only on one varying parameter, but this approach is misleading. The phase space of the system is mixed in the sense that chaotic and regular orbits coexist in the phase space. When we change a parameter of the system, the phase space changes and as a consequence the position of the orbit we suppose to follow changes as well. For example, if we start with an initial setup at which the orbit we examine is chaotic, by changing the spin parameter the orbit with the same otherwise setup will correspond to another trajectory, which might be chaotic or not. Even if we assume that the method of estimating the Lyapunov numbers followed in [1] was correct, then what we see in Figs. 7–13, 15 of [1] is not correlated with the chaoticity of one single orbit, i.e. it cannot provide qualitative

information about the development of the system. If one wanted to do such analysis, the only possible way to follow an orbit through the changing phase space is to track it down in the frequency domain. For regular orbits their characteristic frequencies are their identification numbers. For chaotic orbits the unstable periodic orbits and their corresponding asymptotic manifolds are defining the domain in the phase space which a chaotic orbit covers, so one should track the unstable periodic orbits.

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